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Introduction to electrophysics

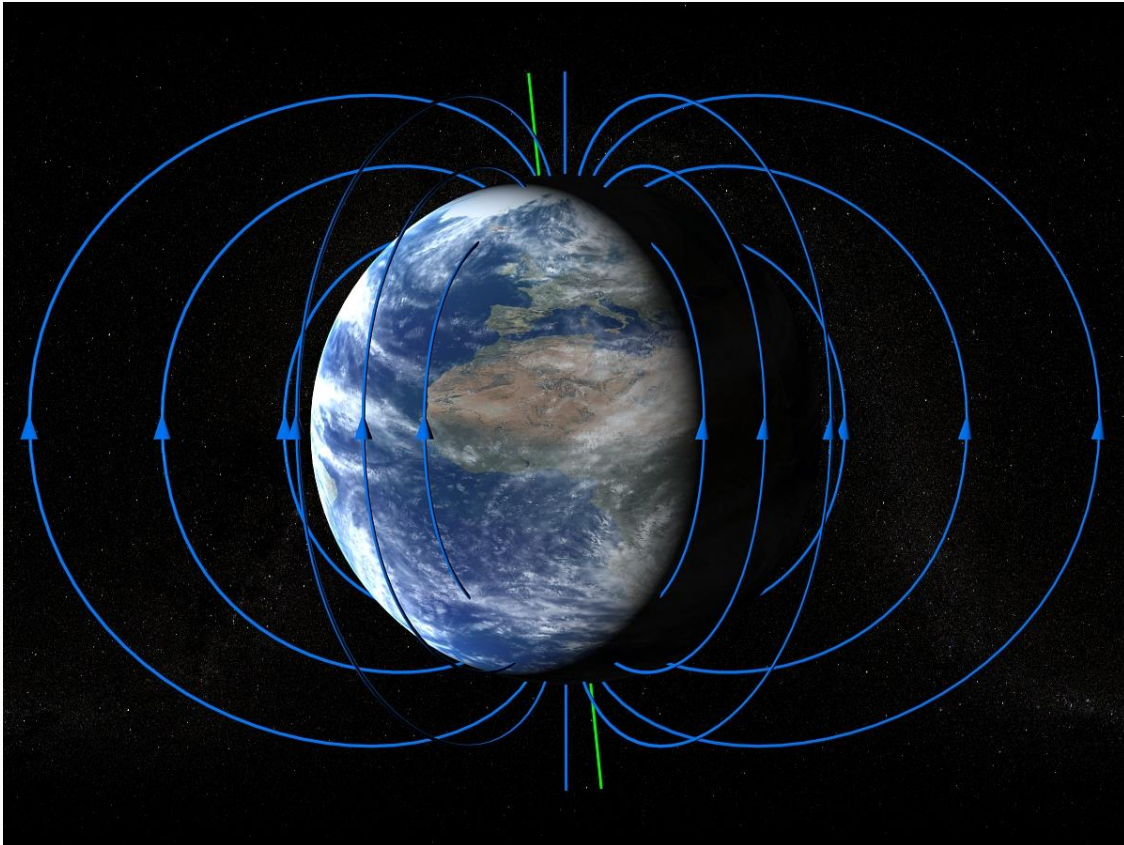


Fig.1.10.1. N.E. Dam ©. The B-field around the earth.

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This first chapter gives (in section 1.1) an overview of the following chapters. The second and third part of this chapter (section 1.2 and section 1.3) introduces some mathematical concepts used in this course, in particular basic vector calculation.

1.1 Overview of the chapters

Chapter 1. Introduction

The first part of this chapter gives a short overview of all the chapters.

The second part of the chapter is a repetition of some of the most used mathematical concepts. Most of all we consider basic definitions concerning vectors such as the scalar product, the vector product, and the triple scalar product, see fig 1.1.1.

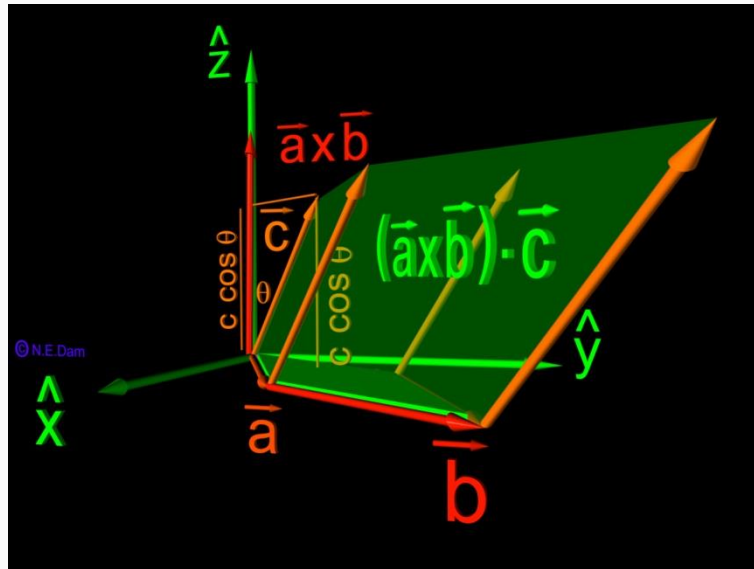


Fig.1.1.1. N.E. Dam ©. Definition of the triple scalar product between two vectors.

Chapter 2. Electric charge, force and field

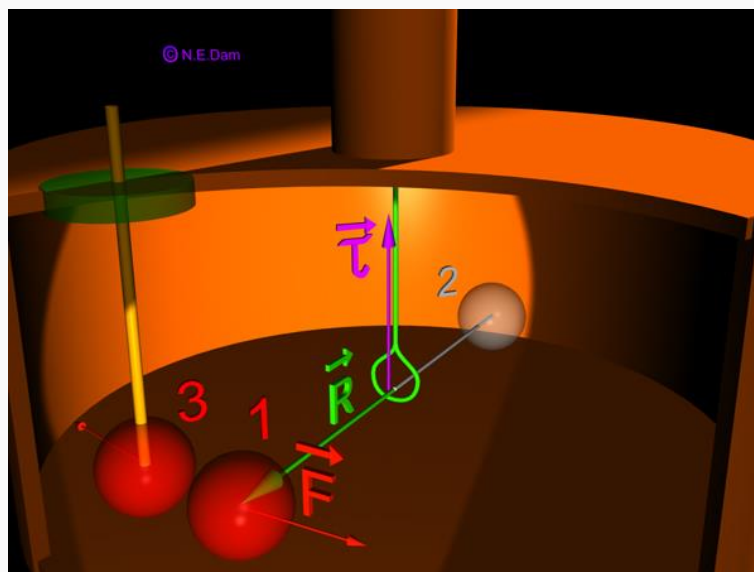


Fig.1.1.2. N.E. Dam ©. A close-up of the equipment used by Coulomb to measure the electric force between two charges.

This chapter starts with the so-called Coulomb force, which is the electric force between two static charges as shown in fig.1.1.2. From this force we define the electric field vector as the force vector per charge unit.

Then we look at a so-called electric dipole. It could be a molecule like water, which is positive at one end and negative at the other end. Such a molecule will be rotated by the electric force until it is parallel with the electric field. We see how to calculate the torque of the dipole. The potential energy of the dipole is calculated in chapter 4.

Chapter 3. Electric flux and charge

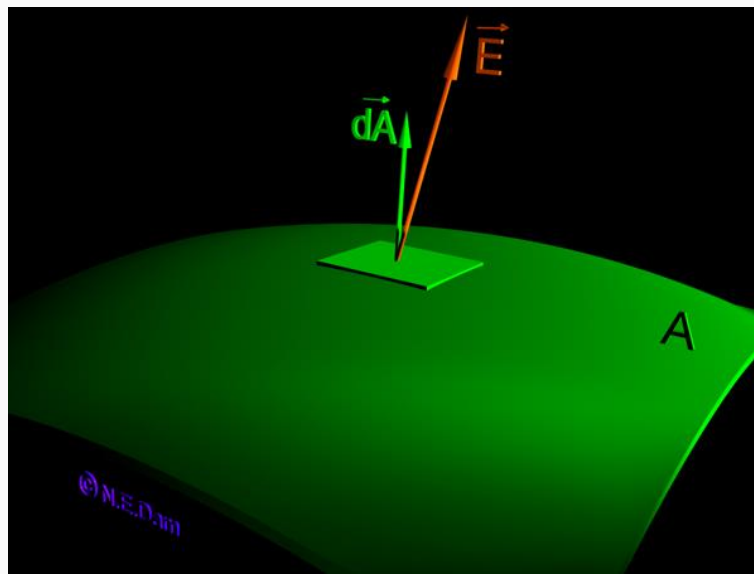


Fig.1.1.3. N.E. Dam ©. Calculation of the electric flux through a curved surface.

In chapter 3 we introduce the very important concept of electric flux. Flux is - in principle - a flow of “something” through a surface. “Something” may be the E-field, and for that case, it will be shown, that the E-flux through an area element dA is the scalar product of the E-field vector and the area vector, $\vec{E} \cdot \vec{dA}$. It should be integrated over a surface to get the flux through that surface, see fig.1.1.3.

Gauss’ law states, that the E-flux out of a closed surface equals the enclosed charge divided by a constant. If the charge distribution has a high symmetry, this law enables us to calculate the electric field strength in a very easy way. The law also enables us to explain many experiments, e.g. the Van der Graaf generator.

Chapter 4. Electric potential energy

If we push a ball uphill to a higher level, we have to overcome the gravitational force, and we know that the ball acquires potential energy. If we push a positive charge q_0

towards another positive charge q , we have to overcome the electric repulsion force, and q_0 acquires electric potential energy, see fig.1.1.4.

In this chapter we see how to calculate the electric potential energy of a charge when the charge is placed in between other charges.

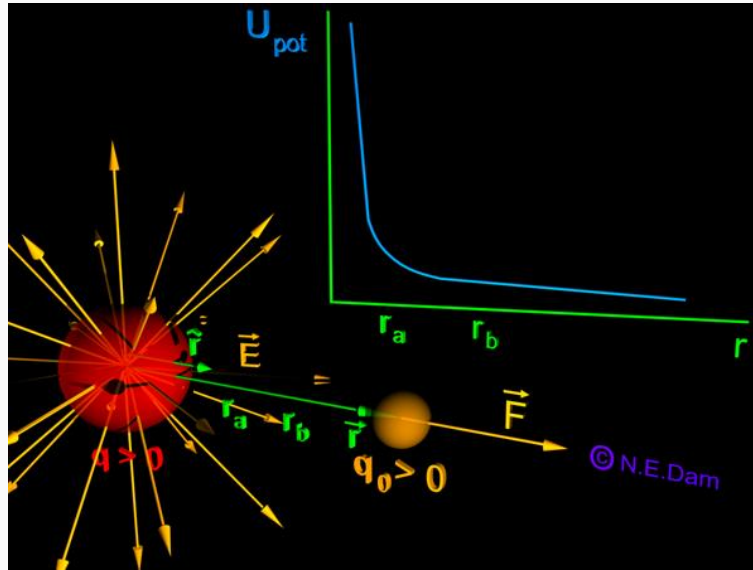


Fig.1.1.4. N.E. Dam ©. Electric potential energy of charge q_0 relative to charge q .

Chapter 5. Electric potential and gradient

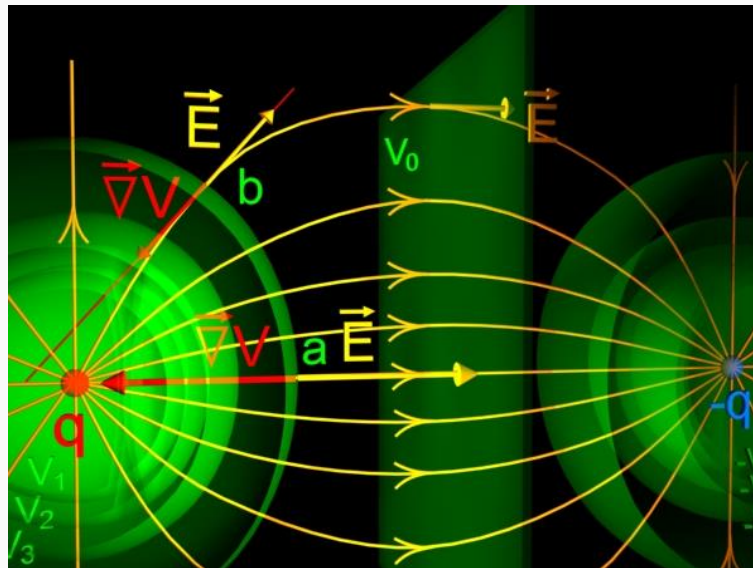


Fig.1.1.5. N.E. Dam ©. Equipotential surfaces around a positive and a negative charge (q and $-q$). The potential is defined as the electric potential energy per unit charge. The potential is visualized as so-called equipotential surfaces perpendicular to the field lines at all points. At all points on an equipotential surface, we have the same potential. Fig.1.1.5. shows (green) equipotential surfaces around a positive and a negative charge (q and $-q$). We see four equipotential surfaces of potential V_1, V_2, V_3 , and V_4 . In this case, the innermost equipotential surface close to q has the highest potential V_1 . The drop in potential from V_1 to V_2 is also called the drop in voltage. In this chapter we see how to calculate the potential at a point surrounded by a distribution of charges.

Chapter 6. Capacitance

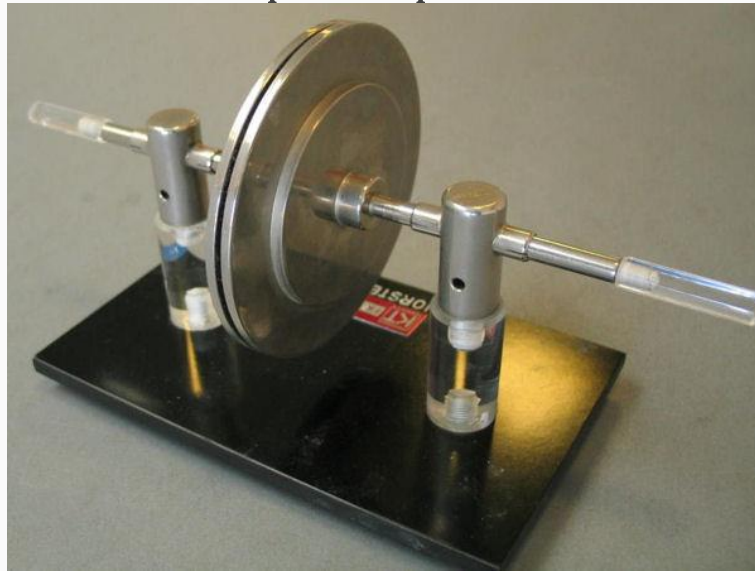


Fig.1.1.6. N.E. Dam ©. A plate capacitor used for demonstration.

This chapter defines the capacitance of a capacitor with and without an insulating material. We will see how to calculate the capacitance for a plate capacitor and for a spherical capacitor. We will also see how to calculate the effect on the capacitance, if we place an insulating material in between the capacitor plates.

Chapter 7. Dielectric materials

The insulator in between the capacitor plates is also called the dielectric material because the molecules act as electric dipoles and orient themselves according to the electric field between the capacitor plates as shown in fig.1.1.7. This effect creates the so-called polarization charge and increases the capacitance significantly.

We will see how the polarization charge creates a so-called polarization field and see how the total electric field in between the capacitor plates is the sum of the external field (the so-called D-field) and the field from the polarization charge (the P-field).

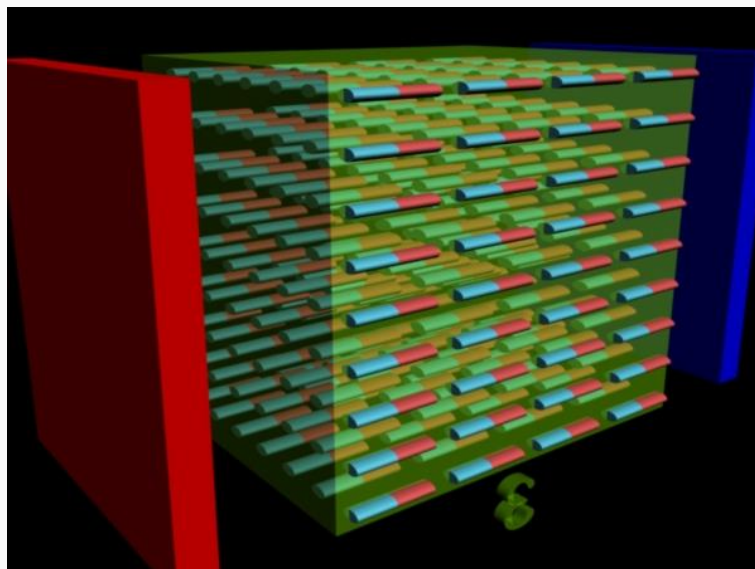


Fig.1.1.7. N.E. Dam ©. A dielectric material in a capacitor.

Chapter 8. Current

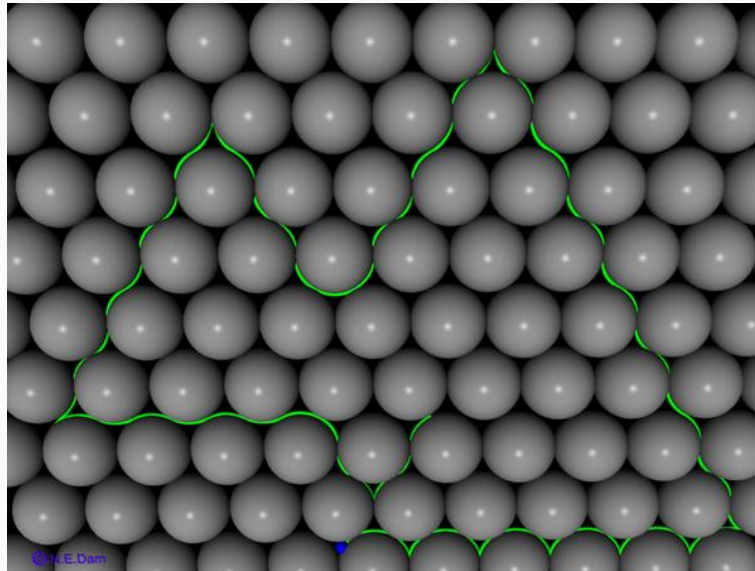


Fig.1.1.8. N.E. Dam ©. An electron moving through a crystal lattice.

At first we look at how electrons are moving inside a conductor, which is typically a metal, see fig.1.1.8. Now and then the conduction electrons collide with other electrons belonging to the atoms in the crystal lattice. Thereby energy is transferred to the crystal lattice, increasing the temperature of the crystal lattice. This energy loss is known as electric resistance. In this chapter we will see the relation between current, potential drop and resistance known as Ohm's law. We also define the resistivity and we see, how the resistivity depends on the temperature for different kinds of materials.

Chapter 9. Elements of electro chemistry

In this chapter we go into detail with the function of a battery. We see, how the different ions move in the electrolyte when the battery is charged and discharged, see fig.1.1.9. We also consider the so-called chemical potential energy and relate this energy to the electric potential difference between the poles of the battery.

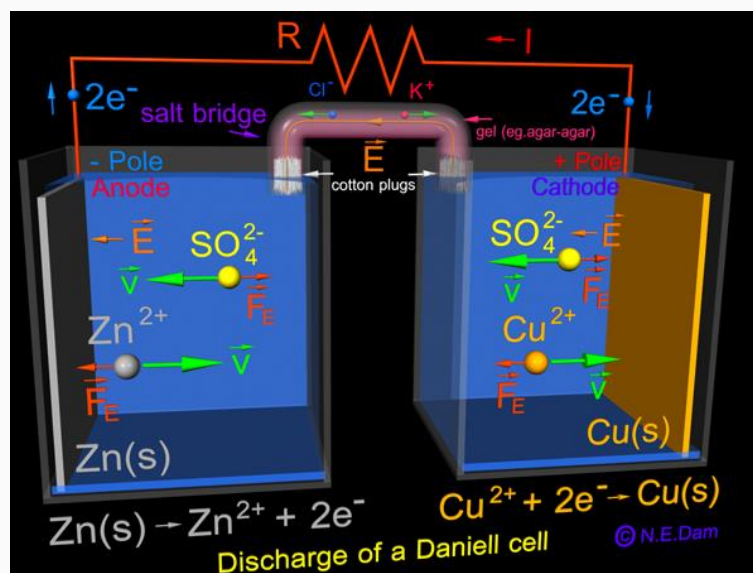


Fig.1.1.9. N.E. Dam ©. The Daniell cell is an example of an electrochemical cell.

Chapter 10. Magnetic sources

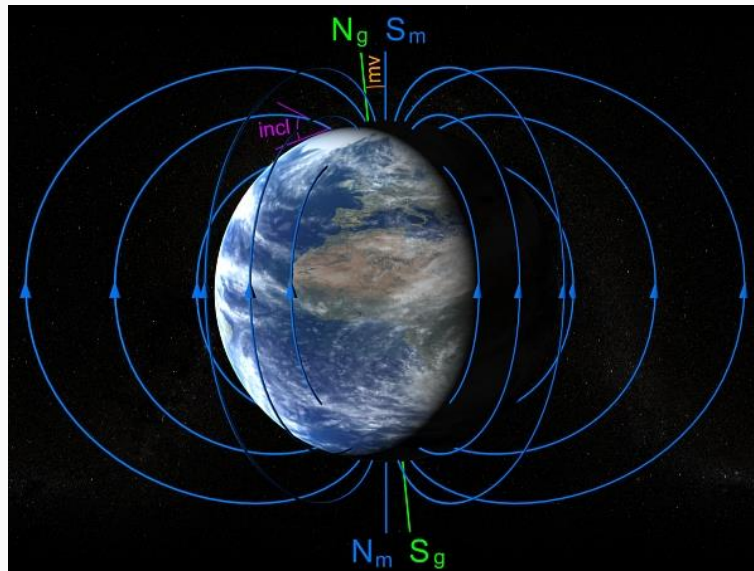


Fig.1.1.10. N.E. Dam ©. The B-field around the earth.

A charge in motion produces a magnetic field (a so-called B-field).

This chapter starts with a description of the earth's field, see fig.1.1.10. Then we will see how to calculate B-fields from different types of sources:

- A single moving charge
- A current flow in a straight wire
- A current flow in a circular loop
- A current flow in a solenoid (a coil)

Coils are used in many different applications. Small coils in electronic devices and relays, large coils in particle accelerators, in hospital scanners and lifting magnets.

Chapter 11. Magnetic force



Fig.1.1.11 (by Mads Pihl), Northern Lights over the backcountry near Sisimiut in Destination Arctic Circle in West Greenland.

A charge must be in motion in a magnetic B-field to be affected by a magnetic force. In this chapter we see how to calculate the magnetic force on a moving charge. In this way charges moving inside a wire will be affected by a magnetic force (a so-called Laplace force). This effect is used in motors, loud speakers and in many other applications. A charge moving in vacuum or in air will also be affected by a magnetic force if there is a B-field. Here, the charge is circling around the B-field lines. This phenomenon is used to generate microwaves used in microwave transmitters (radars, microwave ovens, communication equipment etc.). The phenomenon is also used to explain the magnetosphere and the Northern light, see fig. 1.1.11.

The magnetic force is also used to deflect rays of charged particles in accelerators, in old fashioned TV picture tubes and in mass spectrometers used for chemical analysis.

Chapter 12. Ampere's law

In this chapter we will see a detailed derivation of Ampere's law. The derivation is illustrated with many figures in order to enhance the understanding of the law. The law may seem a bit theoretical. However, it is easy to use in practice. Ampere's law makes it easy to calculate the B-field in cases, where the source of the B-field has a high symmetry (like a wire or a coil).

The chapter ends with a derivation of the expanded version of Ampere's law. Here we see how a change in the E-field may generate a B-field. This is very fundamental for the understanding of electromagnetic waves.

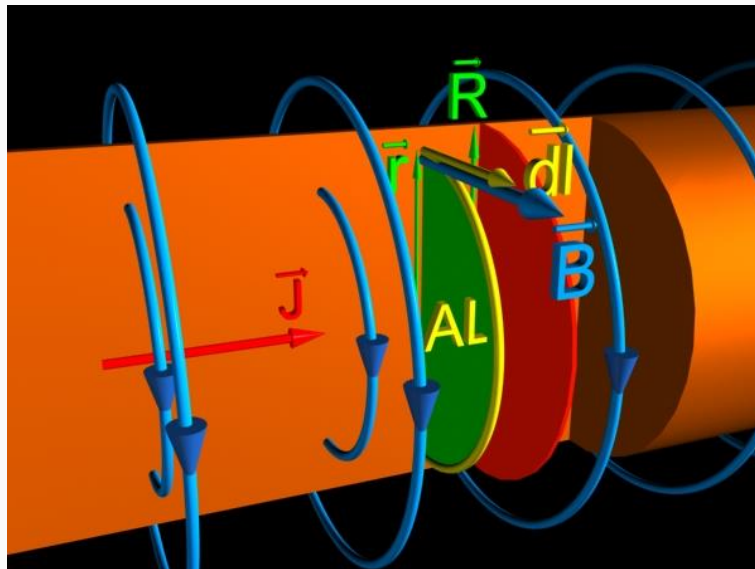


Fig.1.1.12. N.E. Dam ©. Finding the B-field magnitude inside a thick wire using Amperes law.

Chapter 13. Faraday's law of induction

This chapter describes the phenomenon of electromagnetic induction shown in fig.1.1.13, where we have a conducting ring made of a metal. A change in the external magnetic $\vec{B}$ -field through the ring causes an induced circular electric $\vec{E}$ -field in the ring. The induced $\vec{E}$ -field creates an induced current in the ring.

This is used to explain the function of electric generators, transformers and induction brakes.

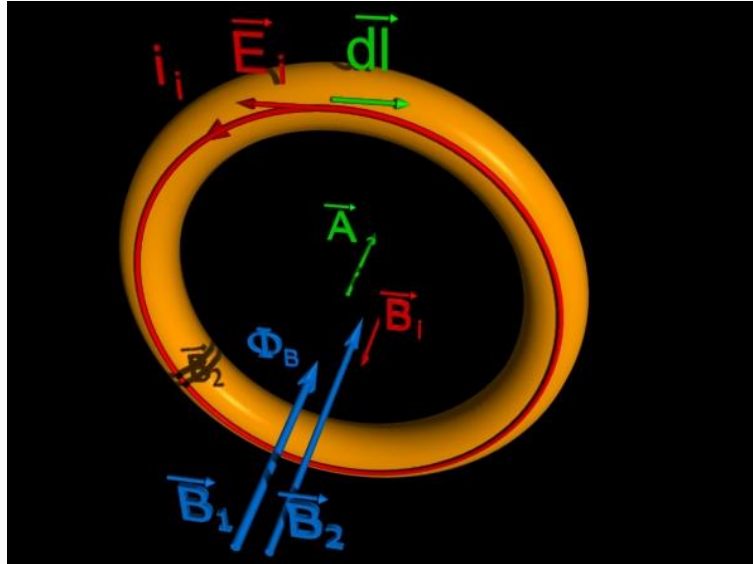


Fig.1.1.13. N.E. Dam ©. The principle of induction.

Chapter 14. Maxwell's equations

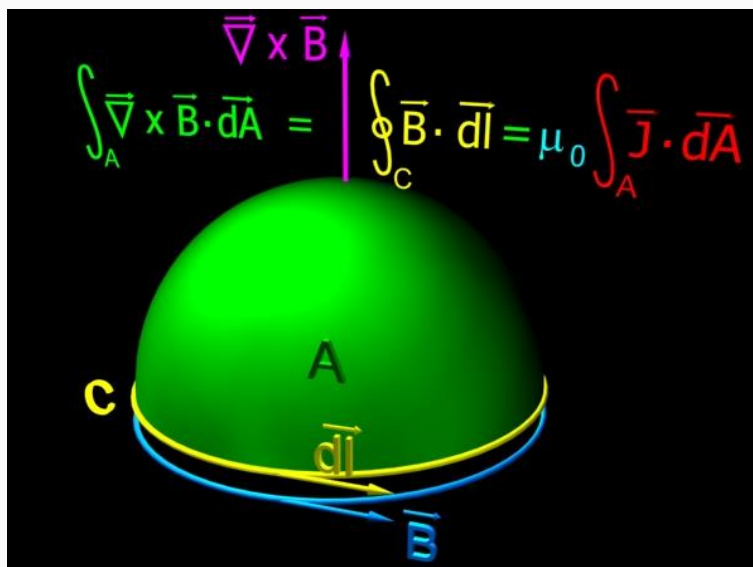


Fig.1.1.14. N.E. Dam ©. Stoke's theorem used in connection with Ampere's law.

The four so-called Maxwell's equations are the fundamental equations of the entire electromagnetism. They are already known from earlier chapters:

1. Gauss's law for the E-field
2. Gauss's law for the B-field
3. Ampere's law
4. Faraday's law of induction

In this chapter we see how the four equations may be changed from the integral form to the differential form, which is widely used in advanced courses of electromagnetism. The differential form may be used to describe waves etc.

Chapter 15. Materials in magnetic fields

This chapter lets you understand in detail what happens when a material is placed in a magnetic B-field. Many atoms act like small magnets or compass needles. We therefore start with a description of how an atom generates a magnetic field. Then we see how the atoms (the small compass needles) align with the external B-field if we place the material in a B-field, see fig.1.1.15. Each atom is equivalent to a circular positive current loop (red) which produces a so-called p-vector, a magnetic dipole moment vector (orange arrow). This p-vector shows the direction of the small B-field produced by the atom. All the p-vectors will try to align with the external B-field just like compass needles. We introduce the so-called M-field for the magnetization, the magnetic field produced by the atoms in the material. We also introduce the H-field for the external field outside the material. The sum of those two fields is the total B-field inside the material.

Magnetic charge does not exist in reality. However, it is convenient to include it in the theory. Then we can re-use much of the relatively simple theory from electric fields and electric charges to describe the more complex magnetic phenomena. Magnetic charge improves the understanding of what happens inside the magnetic material. We will see an analogy to dielectric materials inside a capacitor.

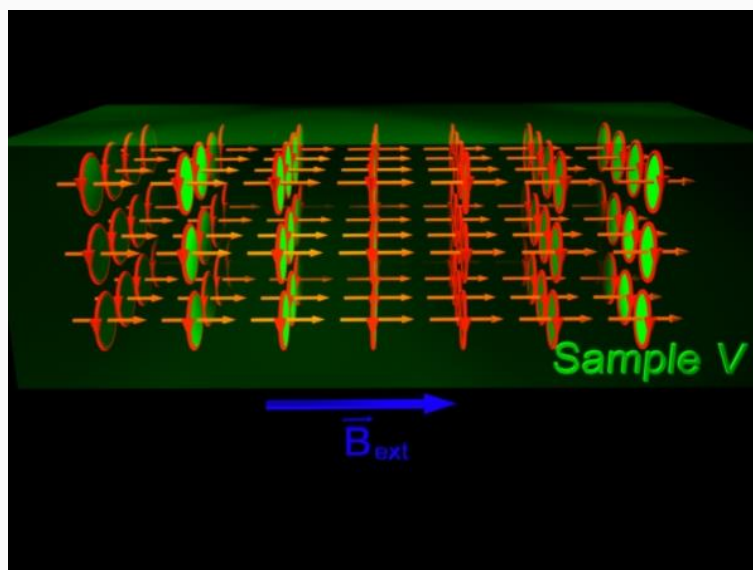


Fig.1.1.15. N.E. Dam ©. When a magnetic material is magnetized, the individual atoms align with the external magnetic B-field.

Chapter 16. Magnetization curves

This chapter is devoted to a very detailed description of the so-called magnetization curves for magnetic materials. Such a curve is normally a closed loop as shown in fig.1.1.16. The curve shows a lot of different properties for a magnetic material. It is used intensively when calculating the B-field inside a magnetic material, which is used when constructing transformers, motors, relays etc. In the beginning of the chapter there is also a description of magnetization curves for non-magnetic materials.

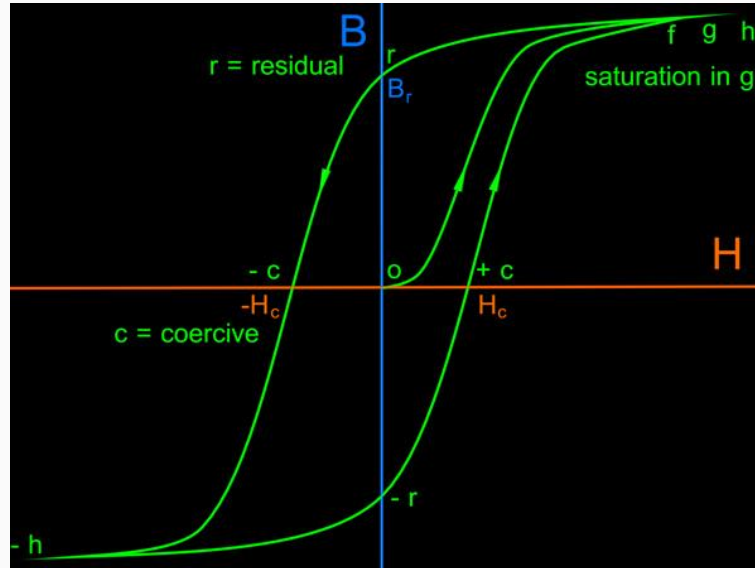


Fig.1.1.16. N.E. Dam ©. $B(H)$ closed loop magnetization curve with hysteresis.

Chapter 17. Magnetic circuits

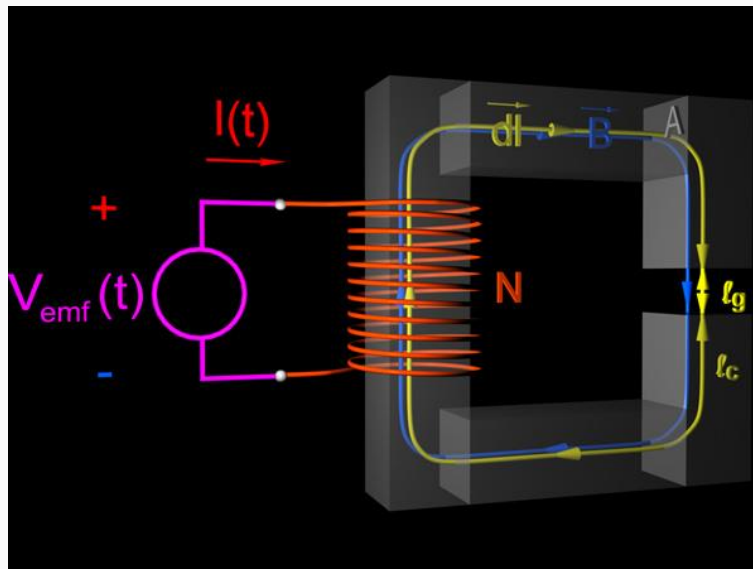


Fig.1.1.17. N.E. Dam ©. An open magnetic circuit with a coil to produce the B-field inside the circuit and an air gap.

This chapter describes magnetic circuits used in relays, transformers, electric magnets and machines.

The chapter describes closed and open magnetic circuits, see fig.1.1.17. We find open magnetic circuits (circuits with an air gap) in relays and electric machines and closed magnetic circuits (without an air gap) in transformers. We will see, that magnetic circuits may be made equivalent to electric circuits, which are easy to calculate.

Chapters 15-17 are beyond the level of most university courses in electrophysics. The subjects of these chapters are usually considered to be rather difficult to understand. In my opinion, it is possible to understand even the most advanced and difficult subjects if they are described carefully in very small steps with very many 3D illustrations. This is what I have tried to do with all the above chapters and especially with chapters 15-17. Many students find chapters 15-17 very interesting. The students achieve a very deep understanding of the magnetic phenomena and learn how to calculate magnetic circuits. The students end up at a rather high level, where they are able to read relative advanced books and articles within electrophysics and magnetism.

1.2 Mathematical concepts used in electrophysics

Definition of an angle

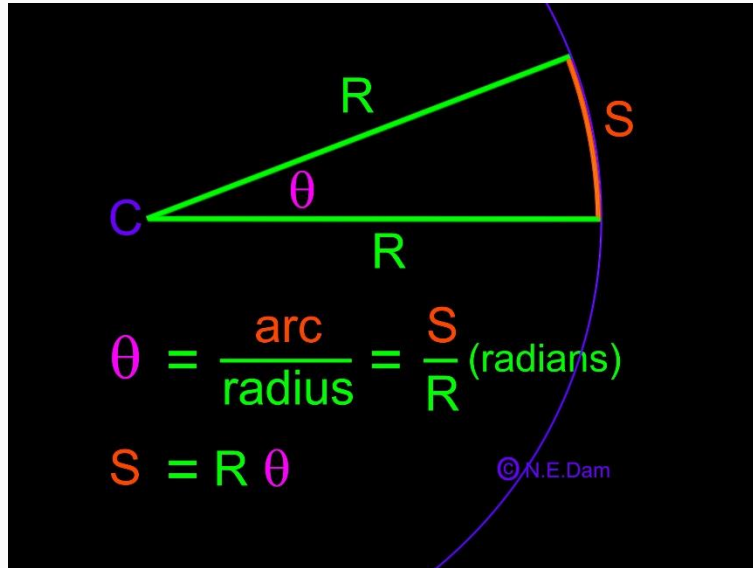


Fig.1.2.1. N.E. Dam ©. Definition of an angle.

The arc S is part of a circle with center C at the acute angle to the left of fig.1.2.1. If you divide the arc S by the radius R of this circle, you get the angle θ in radians.

$$\theta_{\text{radians}} = \frac{S}{R}$$

If you need to convert this angle to degrees, you can divide the angle in radians by π and then multiply by 180 degrees.

$$\theta_{\text{degrees}} = \frac{\theta_{\text{radians}}}{\pi} 180^\circ$$

$\theta_{\text{radians}}/\pi$ is a measure of how big θ_{radians} is relative to half a "turn" in a circle. This fraction is then multiplied by 180 degrees, which corresponds to half a turn.

Definition of cosine, sine and tangent of an angle

Fig.1.2.2 shows the definition of cosine, sine and tangent of an angle in a right-angled triangle. It is recommended that you know these functions and their values at the three angles shown in fig.1.2.3.

It is recommended to consider Exercise 1.1.

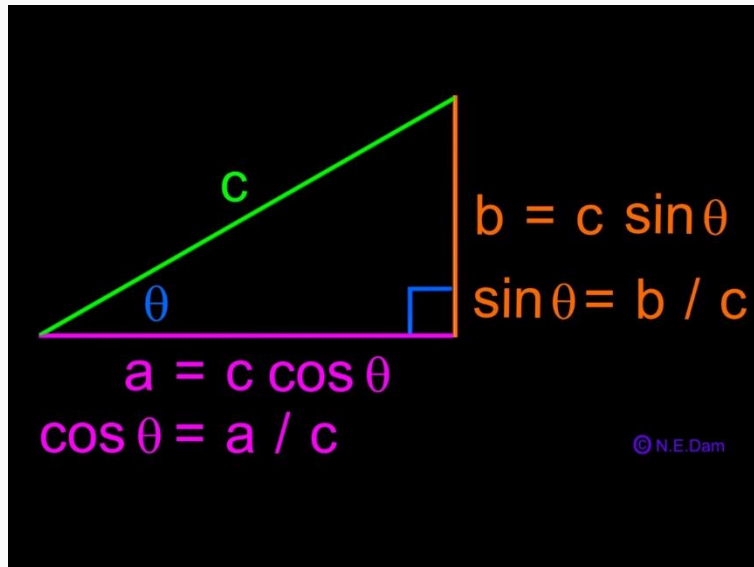


Fig.1.2.2 N.E. Dam ©. Definition of some basic trigonometric functions.

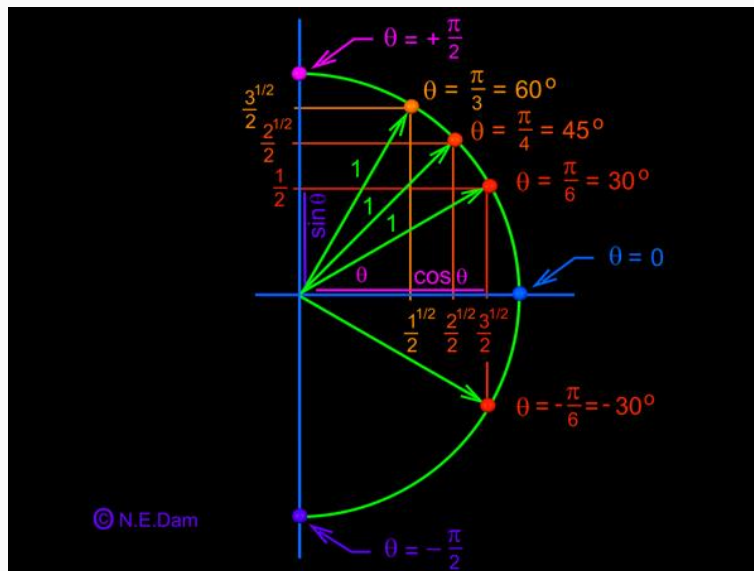


Fig.1.2.3 N.E. Dam ©. Sine and cosine of important angles.

Integration of $1/r^2$

Very often we are confronted with the integration of $1/r^2$, especially when we calculate the electric potential energy from the electric force. We return to the details and the physical explanation in section 4.3. We consider the integral

$$\int \frac{1}{r^2} dr \quad \text{eq. 1.2.1}$$

When we solve this integral we have to guess a function with the property that when we differentiate this function we get $1/r^2$. We guess that this function could be r^{-1} . We know that

$$r^n \leadsto n \cdot r^{n-1} \quad \text{eq. 1.2.2}$$

The $\leadsto$ sign means the differentiation process.

When we differentiate r^{-1} we find

$$r^{-1} \leadsto (-1) \cdot r^{-2} = \frac{-1}{r^2} \quad \text{eq. 1.2.3}$$

This is not quite the same as $1/r^2$. Therefore, we modify our guess to be $-r^{-1}$.

We perform the final differentiation test

$$-r^{-1} \leadsto (-1)(-1) \cdot r^{-2} = \frac{1}{r^2} \quad \text{eq. 1.2.4}$$

Hence, $-r^{-1}$ is right function, i.e. a result of the integration. If we add a constant c to this function and differentiate, we also get $1/r^2$:

$$c - r^{-1} = c - \frac{1}{r} \leadsto 0 + (-1)(-1) \cdot r^{-2} = \frac{1}{r^2} \quad \text{eq. 1.2.5}$$

because $c \leadsto 0$.

The complete solution is therefore

$$\int \frac{1}{r^2} d\mathbf{r} = \mathbf{c} - \frac{1}{\mathbf{r}} \quad \text{eq. 1.2.6}$$

If we consider the same integral with limits we find

$$\int_a^b \frac{1}{r^2} d\mathbf{r} = \left[c - \frac{1}{r} \right]_a^b = \left[c - \frac{1}{b} \right] - \left[c - \frac{1}{a} \right] = \frac{1}{a} - \frac{1}{b} \quad \text{eq. 1.2.7}$$

1.3 Introduction to vectors

Vectors are widely used in physics. A vector may represent a force, a velocity, an electric field, a magnetic field, a magnetization and so on. It is extremely important in electrophysics. Therefore, we repeat the most important concepts below.

Vector coordinates

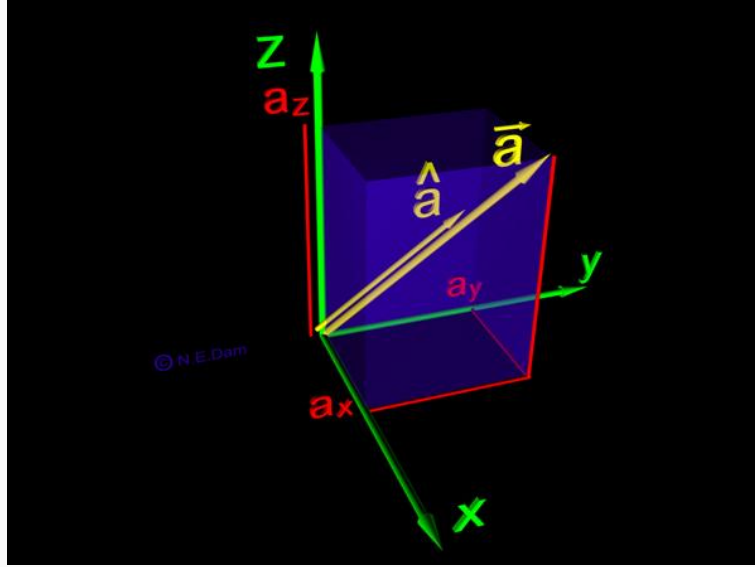


Fig.1.3.1, (N.E. Dam ©). Definition of a vector and the associated unit vector.

A vector may be defined as an arrow (more correctly a mathematical object) with a certain length and direction. The vector may also be defined by its coordinates as shown in fig.1.3.1. The start point of the vector is placed in the origin of a right handed Cartesian coordinate system consisting of the internally perpendicular x, y and z-axis. The coordinates of a vector is the end point coordinates minus the start point coordinates.

$$\vec{a} = \begin{pmatrix} a_{x \text{ end}} \\ a_{y \text{ end}} \\ a_{z \text{ end}} \end{pmatrix} - \begin{pmatrix} a_{x \text{ start}} \\ a_{y \text{ start}} \\ a_{z \text{ start}} \end{pmatrix} \quad \text{eq. 1.3.1}$$

If the start point of the vector is placed in the origin of a coordinate system we see, that the end point of the vector defines the vector

$$\vec{a} = \begin{pmatrix} a_{x \text{ end}} \\ a_{y \text{ end}} \\ a_{z \text{ end}} \end{pmatrix} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \quad \text{eq. 1.3.2}$$

The length of the vector is

$$a = |\vec{a}| = [a_x^2 + a_y^2 + a_z^2]^{\frac{1}{2}} \quad \text{eq. 1.3.3}$$

The unit vector of $\vec{a}$ is defined as

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} \quad \text{eq. 1.3.4}$$

The unit vector $\hat{a}$ has the length 1 and the same direction as $\vec{a}$, see fig.1.3.3.

Note, that $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the unit vectors along the x, y, and z directions.

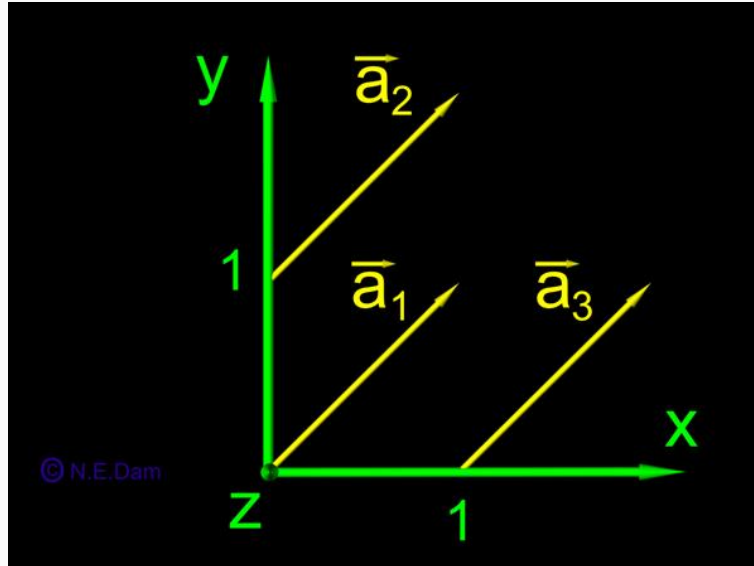


Fig.1.3.2. N.E. Dam ©. Co-parallel vectors of the same length belong to the same class and they are in fact identical.

Direction cosines of a vector

Fig.1.3.3 shows the so-called direction cosine of a vector relative to the x-axis. The blue plane goes through vector $\vec{a}$ and the x-axis. The angle between $\vec{a}$ and the x-axis is θ_x . The cosine of this angle is the x-coordinate of $\vec{a}$ divided by the length of $\vec{a}$

$$\cos(\theta_x) = \frac{a_x}{a} \quad \text{eq. 1.3.5}$$

and likewise for the y and z coordinates:

$$\cos(\theta_y) = \frac{a_y}{a} \quad \text{eq. 1.3.6}$$

$$\cos(\theta_z) = \frac{a_z}{a} \quad \text{eq. 1.3.7}$$

These are the three direction cosines of vector $\vec{a}$.

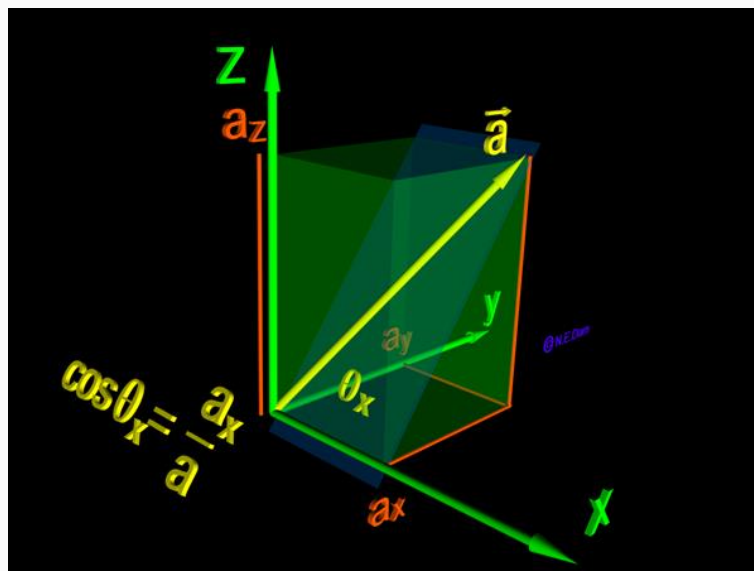


Fig.1.3.3. N.E. Dam ©. Direction cosine relative to the x-axis.

If $\vec{a}$ has the length 1, i.e. if $\vec{a}$ is a unit vector, the three direction cosines are the three coordinates of the vector. In other words, the three direction cosines of a vector $\vec{a}$ are identical with the coordinates of the unit vector $\hat{a}$.

We have for the square sum of the three direction cosines of vector $\vec{a}$

$$(\cos(\theta_x))^2 + (\cos(\theta_y))^2 + (\cos(\theta_z))^2 = \left(\frac{a_x}{a}\right)^2 + \left(\frac{a_y}{a}\right)^2 + \left(\frac{a_z}{a}\right)^2 =$$

$$\frac{a_x^2}{a^2} + \frac{a_y^2}{a^2} + \frac{a_z^2}{a^2} = \frac{a_x^2 + a_y^2 + a_z^2}{a^2} = \frac{a^2}{a^2} = 1 \quad \text{eq. 1.3.8}$$

Eq.1.3.3 defines the length of the vector and eq.1.3.5-7 defines the direction of the vector. Hence, eq.1.3.3 and eq.1.3.5-7 defines the vector completely. Equivalent to this we may define the vector by its coordinates (eq.1.3.1).

Parallel vectors

Coparallel vectors are pointing in the same direction.

Antiparallel vectors are pointing in opposite directions.

Parallel vectors are either coparallel or antiparallel.

The scalar product

Fig.1.3.4 shows the definition of the scalar product of two vectors. The two vectors are placed with their starting points at the origin of the coordinate system. The green plane is placed so it includes the two vectors and the origin. Other names for the *scalar product* are the *inner product* and the *dot product*.

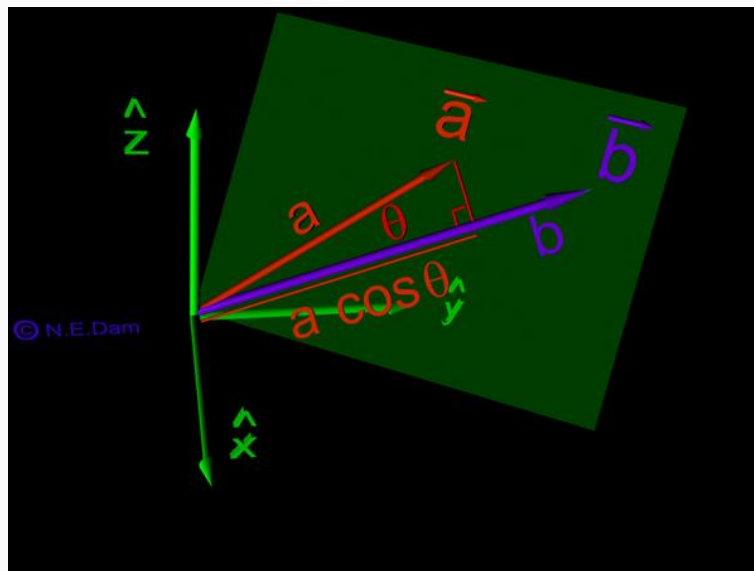


Fig.1.3.4. N.E. Dam ©. Definition of the scalar product between two vectors.

The following is a geometric explanation of the scalar product:

The length of the projection of $\vec{a}$ on $\vec{b}$ is $a \cdot \cos\theta$. This projection is multiplied by the length of $\vec{b}$ to give the scalar product. The geometric explanation is in fact the primary definition of the scalar product:

The scalar product of two vectors is the projection of the first vector along the second vector times the length of the second vector

$$\vec{a} \cdot \vec{b} = (a \cdot \cos\theta) \cdot b = a b \cos\theta \quad \text{eq. 1.3.9}$$

If we use a Cartesian coordinate system, the scalar product of the two vectors $\vec{a}$ and $\vec{b}$ may be defined as follows:

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \cdot \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x b_x + a_y b_y + a_z b_z \quad \text{eq. 1.3.10}$$

Please note, that the scalar product is not a vector. It is a scalar, a number.

If we know the vector coordinates, we can calculate the scalar product $\vec{a} \cdot \vec{b}$ and the magnitudes of the two vectors (a and b) and hence we can use eq.1.3.9 to calculate the angle between the two vectors.

Vector “on the same side”

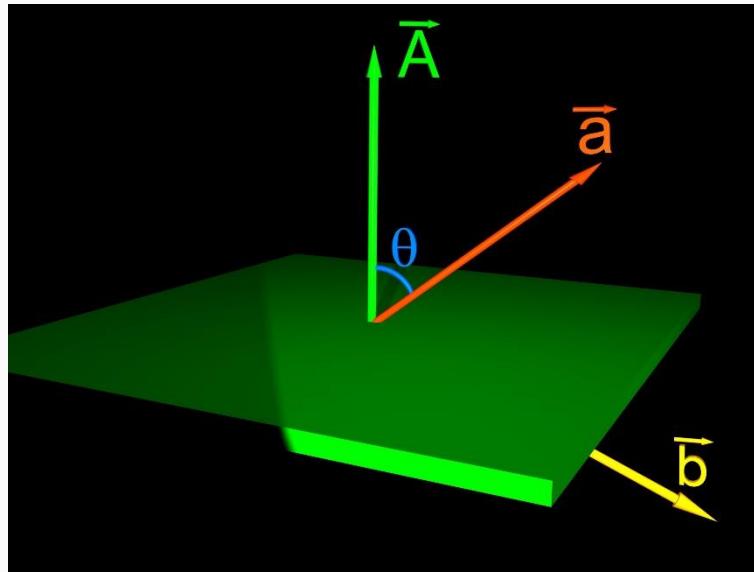


Fig.1.3.5. N.E. Dam ©. Vector $\vec{a}$ is on the same side as $\vec{A}$ i.e. $\vec{a} \cdot \vec{A} > 0$.

Let $\vec{A}$ be a vector which is perpendicular to a plane, see fig.1.3.5.

Vector $\vec{a}$ is said to be on the same side (of the plane) as $\vec{A}$ if the scalar product $\vec{a} \cdot \vec{A}$ is positive, i.e. if the angle θ in the figure is less than 90° .

Vector $\vec{b}$ is on the opposite side of the plane, i.e. the opposite side of $\vec{A}$ if the scalar product $\vec{b} \cdot \vec{A}$ is negative, i.e. if θ in the figure is larger than 90° (and less than 180°).

Vector $\vec{a}$ lies in the plane itself if the scalar product $\vec{a} \cdot \vec{A} = 0$, i.e. if $\vec{a}$ is perpendicular to $\vec{A}$ and $\theta = 90^\circ$.

Area vector

All vectors $\vec{a}$, where $\vec{a} \cdot \vec{A} = 0$, defines a plane perpendicular to vector $\vec{A}$.

If the plane is a limited flat surface as shown in the figure, we may assign a so-called area vector $\vec{A}$ to the plane. The area vector $\vec{A}$ is perpendicular to the plane and its length is a measure of the area A of the plane.

It is recommended to consider Exercise 1.2.

The vector product

Fig.1.3.6 shows the vector product of two vectors. Here we have the two vectors $\vec{a}$ and $\vec{b}$ in the green plane. The vector product $\vec{a} \times \vec{b}$ is a vector, that is perpendicular to both $\vec{a}$ and $\vec{b}$. Since $\vec{a}$ and $\vec{b}$ are in the green plane, the vector product $\vec{a} \times \vec{b}$ is also perpendicular to the green plane.

The direction of the vector product $\vec{a} \times \vec{b}$ is found by using our right hand.

Take your right hand fingers along vector $\vec{a}$ and bend the fingers along $\vec{b}$. The right hand thumb will then show the direction of the vector product $\vec{a} \times \vec{b}$.

The length or magnitude of the vector product $\vec{a} \times \vec{b}$ is the area of the parallelogram defined by $\vec{a}$ and $\vec{b}$ shown as the light green area in the figure. This area is the baseline a times the height of parallelogram, which is

$b \cdot \sin\theta$, where θ is the angle from $\vec{a}$ to $\vec{b}$.

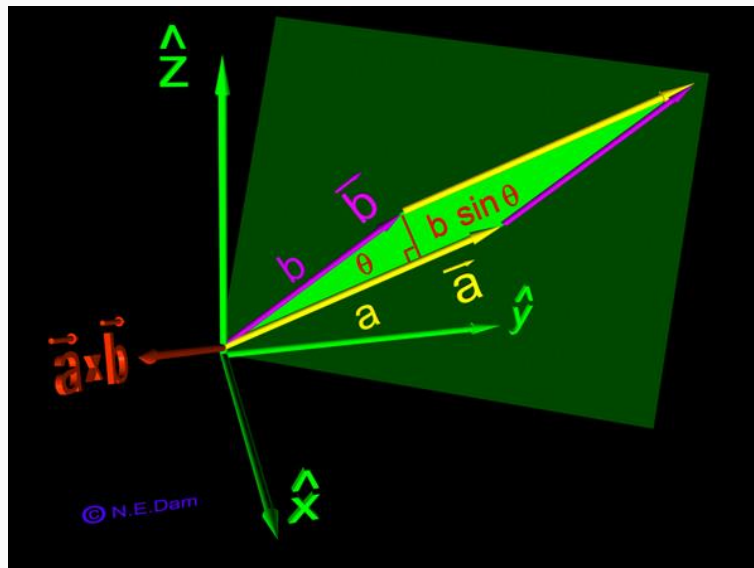


Fig.1.3.6. N.E. Dam ©. Definition of the vector product of two vectors

The geometric explanation

The vector product $\vec{a} \times \vec{b}$ is a vector, that is perpendicular to $\vec{a}$ and $\vec{b}$.

The direction is along our right hand thumb, when we take our fingers along first $\vec{a}$ and then $\vec{b}$.

The length or magnitude of the vector product is the area of the parallelogram defined by $\vec{a}$ and $\vec{b}$.

The length or magnitude of the vector product is

$$|\vec{a} \times \vec{b}| = a (b \cdot \sin\theta) = ab \sin\theta, \text{ where } \theta = \angle(\vec{a}, \vec{b}) \quad \text{eq. 1.3.11}$$

θ is the angle *from* $\vec{a}$ *to* $\vec{b}$.

The mathematical definition

Using coordinates, *the mathematical definition* the vector product is

$$\vec{a} \times \vec{b} = (a_x, a_y, a_z) \times (b_x, b_y, b_z) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \quad \text{eq. 1.3.12}$$

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x) \quad \text{eq. 1.3.13}$$

It is recommended to consider Exercise 1.3.

The triple scalar product

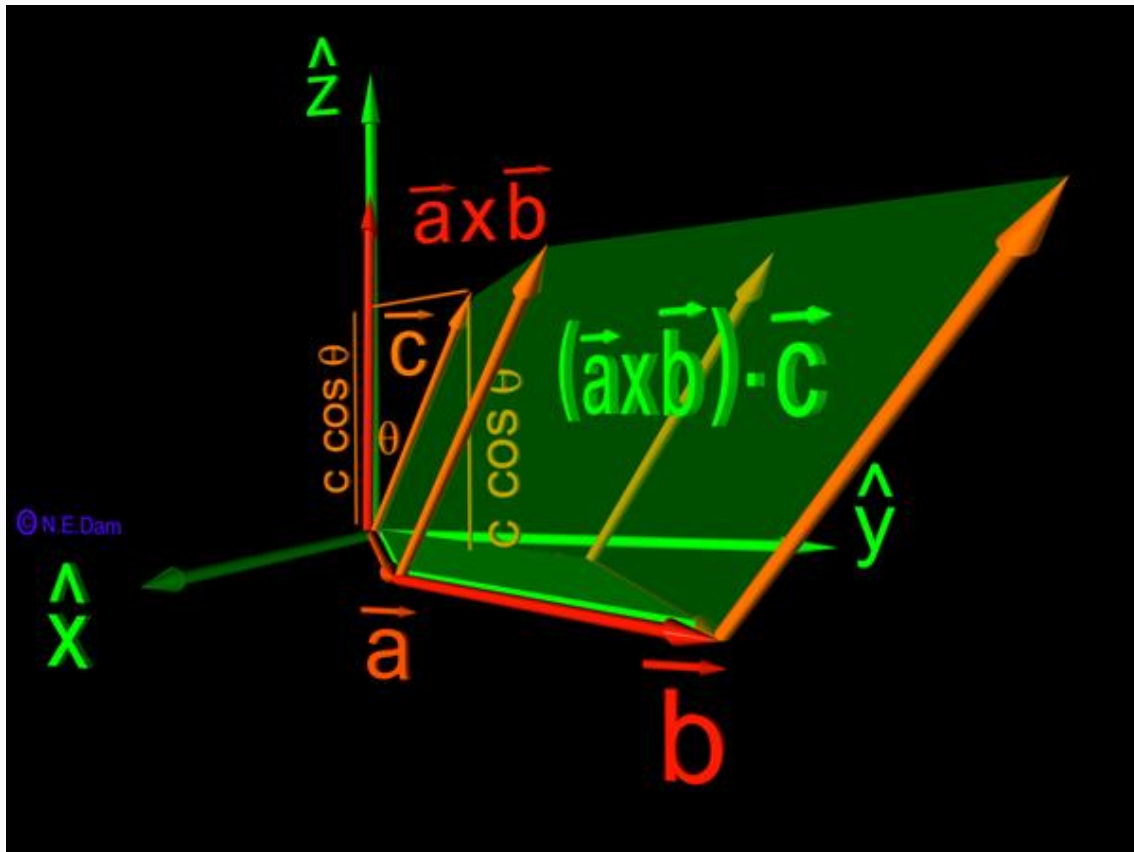


Fig.1.3.7. N.E. Dam ©. The triple scalar product is the volume of the parallelepiped defined by the three vectors $\vec{a}, \vec{b}, \vec{c}$

The triple scalar product of the three vectors $\vec{a}, \vec{b}, \vec{c}$ is defined as

$$[\vec{a}, \vec{b}, \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c} \quad \text{eq. 1.3.14}$$

The triple scalar product is called the “space product” in Danish (Danish “rum produkt”).

The reason may be, that the triple scalar product is the volume of the *parallelepiped* defined by the three vectors $\vec{a}, \vec{b}, \vec{c}$ in the product, see fig.1.3.7.

According to the previous section, the magnitude of the vector product $\vec{a} \times \vec{b}$ is the area of the parallelogram defined by the two vectors $\vec{a}, \vec{b}$, see also fig.1.3.6.

Considering the triple scalar product we have

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = |\vec{a} \times \vec{b}| |\vec{c}| \cos(\theta), \text{ where } \theta = \angle(\vec{a} \times \vec{b}, \vec{c}) \quad \text{eq. 1.3.15}$$

Here $|\vec{c}| \cos(\theta)$ is the height of the parallelepiped shown in fig.1.3.7. It explains why the triple scalar product is the volume of the parallelepiped.

If two of the three vectors in the triple scalar product are parallel, the volume of the parallelepiped is zero.

If all three vectors in the triple scalar product are in the same plane, then for example vector $\vec{c}$ is perpendicular to $\vec{a} \times \vec{b}$ and hence the volume is zero.

We may interchange the dot and the cross product in the triple scalar product without changing the triple scalar product

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}) \quad \text{eq. 1.3.16}$$

because it gives the same volume.

The triple scalar product may also be written as

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{c} \times \vec{a}) \cdot \vec{b} = (\vec{b} \times \vec{c}) \cdot \vec{a} \quad \text{eq. 1.3.17}$$

with the same cyclic order of the three vectors.

However, if we interchange two vectors, the sign of the triple scalar product will change

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = -(\vec{b} \times \vec{a}) \cdot \vec{c} \quad \text{eq. 1.3.18}$$

In other words we have for the triple scalar product:

We can interchange $\times$ and $\cdot$ and we can displace the vectors in $[\vec{a}, \vec{b}, \vec{c}]$ cyclic without changing the triple scalar product. However, if we interchange two vectors, the triple scalar product changes its sign.

The triple scalar product may be written as

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \quad \text{eq. 1.3.19}$$

It is recommended to consider Exercise 1.4.

Exercises

Exercise 1.1

Consider a right angle triangle with the hypotenuse $R = 2$ and the angle $\Theta = \pi/6$ as shown in fig.1.2.1.

1. Make a sketch of the triangle
2. Find the cathetes x and y
3. Find the arc S

Exercise 1.2

Let vector $\vec{a} = (1 \ 1 \ 0)$ and $\vec{b} = (0 \ 1 \ 0)$.

Find $\vec{a} \cdot \vec{b}$ and $\cos\theta$

Make a drawing.

Exercise 1.3

Let vector $\vec{a} = (1 \ 1 \ 0)$ and $\vec{b} = (0 \ 1 \ 0)$.

Find $|\vec{a} \times \vec{b}|$ and $\vec{a} \times \vec{b}$

Make a drawing.

Exercise 1.4

Let vector $\vec{a} = (1 \ 1 \ 0)$, $\vec{b} = (0 \ 1 \ 0)$ and $\vec{c} = (1 \ 1 \ 1)$.

Find $[\vec{a}, \vec{b}, \vec{c}]$

Make a drawing.